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TILINGS OF THE PLANE: THURSTON SEMI-NORM AND DECIDABILITY

JEAN-RENÉ CHAZOTTES, JEAN-MARC GAMBAUDO, AND FRANÇOIS GAUTERO

Abstract. We give a geometric interpretation of the undecidability of the tiling problem in the plane. We show that this problem boils down to prove the existence of zeros of a nonnegative convex function defined on a finite-dimensional simplex and related to the Thurston semi-norm.

1. Introduction

Let us consider a finite collection $\mathscr{P} = \{p_1, \ldots, p_n\}$, where for $j = 1, \ldots, n$, p_j is a polygon in the Euclidean plane $\mathbb{R}^2$, indexed by j and with colored edges. These decorated polygons are called *prototiles*. In the sequel a *rational prototile* (resp. *integral prototile*) is a prototile whose vertices have rational (resp. integer) coordinates. A tiling of $\mathbb{R}^2$ made with $\mathscr{P}$ is a collection $(t_i)_{i\geqslant 0}$ of polygons called *tiles* indexed by a symbol $k(i)$ in $\{1, \ldots, n\}$, such that:

- the tiles cover the plane: $\cup_{i\geqslant 0} t_i = \mathbb{R}^2$;
- the tiles have disjoint interiors: $\mathrm{int}(t_i) \cap \mathrm{int}(t_j) = \varnothing$ whenever $i \neq j$;
- whenever two distinct tiles intersect, they do it along a common edge and the colors match;
- for each $i \geqslant 0$, t_i is a translated copy of $p_{k(i)}$.

We denote by $\Omega_{\mathscr{P}}$ the set of all tilings made with $\mathscr{P}$. Clearly this set may be empty. When $\Omega_{\mathscr{P}} \neq \varnothing$ the group of translation acts on $\Omega_{\mathscr{P}}$ as follows:

$$\Omega_{\mathscr{P}} \times \mathbb{R}^2 \ni (T, u) \mapsto T + u \in \Omega_{\mathscr{P}}$$

where $T + u = (t_i + u)_{i\geqslant 0}$ whenever $T = (t_i)_{i\geqslant 0}$. A tiling $T \in \Omega_{\mathscr{P}}$ is *periodic* if there exist two independent vectors u_1 and u_2 in $\mathbb{R}^2$ such that $T = T + u_1 = T + u_2$. R. Berger [4] proved that the problem to know whether or not $\Omega_{\mathscr{P}}$ is empty (*i.e* if one can or cannot tile the plane with $\mathscr{P}$) is not decidable. More precisely he showed that there is no algorithm that can take as input any family of prototiles $\mathscr{P}$ and give as output in a finite time one of the following two results: $\mathscr{P}$ can tile the plane or $\mathscr{P}$ cannot tile the plane. Berger also showed that this undecidability is strongly related to the fact that there exist collections of prototiles $\mathscr{P}$ which tile the plane ($\Omega_{\mathscr{P}} \neq \varnothing$) but not periodically.

The aim of this paper is to give a geometric interpretation of this undecidability property.

Before we state our main result, we need some notations and definitions.

The Anderson-Putnam CW-complex associated with a collection $\mathscr{P}$ of prototiles is

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the cell complex $AP_{\mathscr{P}}$ (see [1]) [1] made with 2-cells, 1-cells and 0-cells constructed as follows. There is one 2-cell for each prototile and these 2-cells are glued along their colored edges by translation [1]. An edge e_{k_0} of p_{k_0} is glued to an edge e_{k_1} of p_{k_1} if and only if :

- they have the same color;
- there exists a vector v_{k_0,k_1} in $\mathbb{R}^2$ such that $e_{k_1} = e_{k_0} + v_{k_0,k_1}$.

For $i = 1, 2$, the vector space of linear combinations with real coefficients of the oriented i-cells is denoted by $C_i(AP_{\mathscr{P}}, \mathbb{R})$, its elements are called i-chains and the coefficients are called coordinates. For any chain c in $C_i(AP_{\mathscr{P}}, \mathbb{R})$, we denote by $|c|$ its ℓ_1-norm. By convention, for each i-chain c , $-c$ is the chain which corresponds to an inversion of the orientation. Notice that there is a natural orientation of the 2-cells induced by the orientation of $\mathbb{R}^2$, but there is no natural orientation of the 1-cells. Given an arbitrary orientation on each 1-cell of $AP_{\mathscr{P}}$, the 2-cells that contain this edge are split in two parts: the positive ones for which the orientation on the edge coincides with the one induced by the orientation of the 2-cell and the negative ones for which both orientations are different. Notice that this splitting is independent, up to reversing, on the arbitrary choice of the orientation of the 1-cells.

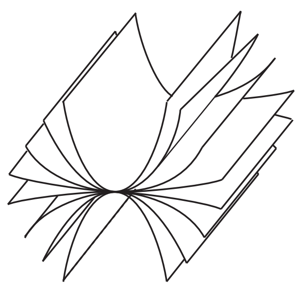

FIGURE 1. Local view of the Anderson-Putnam complex.

We define the linear boundary operator

$$\partial : C_2(AP_{\mathscr{P}}, \mathbb{R}) \to C_1(AP_{\mathscr{P}}, \mathbb{R})$$

which assigns to any face the sum of the edges at its boundary, weighted with a positive (resp. negative) sign if the induced orientation fits (resp. does not fit) with the orientation chosen for these edges. The kernel of the operator ∂ is the vector space of 2-cycles which we denote $H_2(AP_{\mathscr{P}}, \mathbb{R})$. It is well known that (up to an isomorphism) the vector space $H_2(AP_{\mathscr{P}}, \mathbb{R})$ is a topological invariant of $AP_{\mathscr{P}}$ that coincides with the second singular homology group of the branched surface $AP_{\mathscr{P}}$ (see for example [13]). The canonical orientation of the faces allows us to characterize the vector space $H_2(AP_{\mathscr{P}}, \mathbb{R}) \subset C_2(AP_{\mathscr{P}}, \mathbb{R})$ as follows. A 2-chain is

[1]Actually the construction given by Anderson and Putnam is made in a particular case that suppose that $\mathscr{P}$ tiles the plane. They get a cell complex which is smaller than the one we defined here, however the basic ideas of both constructions are the same.

a 2-cycle if and only if for each edge e the sum of the coordinates of the positive faces containing e is equal to the sum of the coordinates of the negative faces. This gives a set of m linear equations with integer coefficients for n variables (where m is the dimension of $C_1(AP_{\mathscr{P}}, \mathbb{R})$ and n the dimension of $C_2(AP_{\mathscr{P}}, \mathbb{R})$). These equations are called the *switching rules*. Let us say that a 2-cycle is *nonnegative* if its coordinates are greater than or equal to zero and denote by $H_2^+(AP_{\mathscr{P}}, \mathbb{R})$, the closed cone of nonnegative cycles, *i.e* the closed cone of cycles with nonnegative coordinates and by $S_2(AP_{\mathscr{P}}, \mathbb{R})$ the simplex made of all non negative cycles whose sum of their coordinates is 1. Finally, let us say that a 2-cycle is integral (resp. rational) if its coordinates are integers (resp. rational numbers).

When $\Omega_{\mathscr{P}} \neq \emptyset$, *i.e* when one can tile $\mathbb{R}^2$ with $\mathscr{P}$, $\Omega_{\mathscr{P}}$ inherits a natural metrizable topology. A metric δ defining this topology can be chosen as follows. Let $B_\epsilon(0)$ stand for the open ball with radius ϵ centered at 0 in $\mathbb{R}^d$ and $B_\epsilon[T] := \{t_j \in T : t_j \cap \bar{B}_\epsilon(0) \neq \emptyset\}$ be the collection of tiles in T that meet $\bar{B}_\epsilon(0)$. Consider in $\Omega_{\mathscr{P}}$ two tilings T and T' and let A denote the set of ϵ in $(0,1)$ such that there exists u and u' in $\mathbb{R}^d$, with $|u|, |u'| \leqslant \epsilon/2$, so that $B_{1/\epsilon}[T+u] = B_{1/\epsilon}[T'+u']$. Then :

$$\delta(T,T') = \begin{cases} \inf A & \text{if } A \neq \emptyset \\ 1 & \text{if not.} \end{cases}$$

In words T and T' are close if, up to a small translation, they agree exactly in a large neighborhood of the origin. Equipped with such a metric, $\Omega_{\mathscr{P}}$ is a compact metric space and the $\mathbb{R}^2$ action by translation is continuous. Since the group $\mathbb{R}^2$ is amenable, it follows that the dynamical system $(\Omega_{\mathscr{P}}, \mathbb{R}^2)$ possesses finite translation-invariant measures and we denote by $\mathcal{M}(\Omega_{\mathscr{P}})$ the set of finite translation-invariant measures on $\Omega_{\mathscr{P}}$ and by $\Theta(\Omega_{\mathscr{P}})$ the subset of $\mathcal{M}(\Omega_{\mathscr{P}})$ made of probability measures. There exists a natural morphism:

$$\mathrm{Ev} : \mathcal{M}(\Omega_{\mathscr{P}}) \to C_2(AP_{\mathscr{P}}, \mathbb{R})$$

defined by:

$$\mathrm{Ev}(\mu) = \sum_{i=1}^{i=n} \frac{\mu(\pi^{-1}(p_i))}{\lambda(p_i)}\, p_i$$

where λ stands for the Lebesgue measure in $\mathbb{R}^2$ and $\pi : \Omega_{\mathscr{P}} \to AP(\mathscr{P})$ is the natural projection which associates to each tiling the location of the origin of $\mathbb{R}^2$ in the (translated copy of the) prototile where it belongs. The switching rules reflect the translation invariance and it comes easily that:

Proposition 1.1. [3]

$$\mathrm{Ev}(\mathcal{M}(\Omega_{\mathscr{P}})) \subset H_2^+(AP_{\mathscr{P}}, \mathbb{R}) \quad \text{and} \quad \mathrm{Ev}(\Theta(\Omega_{\mathscr{P}})) \subset S_2(AP_{\mathscr{P}}, \mathbb{R}).$$

It turns out that the above inclusions may not be onto: one can find a set of prototiles and non negative coefficients associated with these prototiles satisfying the switching rules which are not the weights of some finite translation invariant measure (see [6]). Clearly the set $\mathrm{Ev}(\Theta(\Omega_{\mathscr{P}}))$ is a closed convex subset of $S_2(AP_{\mathscr{P}}, \mathbb{R})$, and the aim of this paper is to characterize this subset, that is to say to give geometric conditions on the non negative 2-cycles insuring that they are images of finite measures. We will construct a non negative continuous convex function $S_2(AP_{\mathscr{P}}, \mathbb{R}) \ni c \mapsto |||c||| \in \mathbb{R}$ called the *asymptotic Thurston semi-norm* (see [14] for a first description of this norm) and show the following theorem which is our main result:

Theorem 1.1.
A collection of prototiles $\mathscr{P}$ tiles the plane if and only if $H_2^+(AP_{\mathscr{P}}, \mathbb{R}) \neq \emptyset$ and the asymptotic Thurston semi-norm has a zero in $S_2(AP_{\mathscr{P}}, \mathbb{R})$. In this case

$$c \in \mathrm{Ev}(\Theta(\Omega_{\mathscr{P}})) \Longleftrightarrow |||c||| = 0.$$

2. The asymptotic Thurston semi-norm

For each non negative integral cycle $c \in H_2^+(AP_{\mathscr{P}}; \mathbb{R})$, we denote by $[c]$ the set of all compact oriented surfaces $\mathcal{F}$ that are covers of the flat branched surface $AP_{\mathscr{P}}$ and whose associated 2-cycles $c_{\mathcal{F}}$ is equal to c. In [7] it is proved that this set is not empty and that these surfaces are naturally equipped with a translation structure[2]. Associated with any non negative integral cycle, let us consider the following quantity:

$$\|c\| \equiv -\max_{\mathcal{F}\in[c]}(\chi(\mathcal{F})),$$

where $\chi(.)$ stands for the Euler characteristic[3]. For any integer $n > 0$ and any pair of non negative integral cycles, c and c', we have:

$$\|nc\| \leqslant n\|c\|, \text{ and } \|c + c'\| \leqslant \|c\| + \|c'\|.$$

Remark 2.1. *For any non negative integral cycle c:*

$$\|c\| = 0 \Longleftrightarrow [c] \text{ contains a torus.}$$

It follows that for any non negative integral cycle c, the limit of the sequence $((1/n)\|n.c\|)_n$ exists. We define:

$$|||c||| \equiv \lim_{n\to+\infty} \frac{1}{n}\|nc\|.$$

We observe that for each integer $n > 0$, and each pair of non negative integer classes c and c', we have:

$$|||nc||| = n|||c|||, \quad \text{and} \quad |||c + c'||| \leqslant |||c||| + |||c'|||.$$

Thus, we can extend the definition of the asymptotic Thurston semi-norm $|||\cdot|||$ to the rational cycles in $P_2(AP_{\mathscr{P}}; \mathbb{R})$, by setting :

$$|||c||| \equiv \frac{1}{n}|||nc|||,$$

for all integer n such that $n.c$ is an integral cycle.

Remark 2.2. $|||c||| = 0$ *does not imply that $[c]$ contains a torus. Compare with Remark 2.1.*

Lemma 2.3. *There exists a constant $C_{\mathscr{P}} > 0$ such that for any rational cycle in $S_2(AP_{\mathscr{P}}, \mathbb{R})$ (resp. in $H_2^+(AP_{\mathscr{P}}, \mathbb{R})$), we have:*

$$|||c||| \leqslant C_{\mathscr{P}} \quad (\text{resp. } |||c||| \leqslant C_{\mathscr{P}}.|c|).$$

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[2] In particular it has a meaning to move vertically or horizontally from any point (except a finite number) of the surface.

[3] We recall that the Euler characteristic of a surface is the sum of the Euler characteristics of its connected component.

Proof. Let us observe first that the extreme points of the simplex $S_2(AP_{\mathscr{P}}, \mathbb{R})$ belong to the boundary of the simplex

$$S = \Big\{(a_1, \ldots, a_n) \in \mathbb{R}^n, \quad \text{with} \quad \sum_{i=1}^{i=n} a_i = 1 \quad \text{and} \quad a_i \geqslant 0, \forall i \in \{1, \ldots, n\}\Big\}.$$

Let $e_1, \ldots, e_k$ be these extreme points ($k \leqslant n$). Since each of these extreme points is the unique solution of a system of linear equations with integer coefficients, their coefficients are rational cycles. For $j = 1, \ldots, k$, let n_j be the smallest integer such that $\bar{e}_j = n_j e_j$ is an integral cycle. Let $\mathscr{L}^+$ be the restriction to the positive quadrant of the lattice in $\mathbb{R}^n$ generated by the cycles $\bar{e}_1, \ldots, \bar{e}_k$, *i.e* the cycles that read:

$$l = \sum_{j=1}^{j=k} k_j \bar{e}_j \text{ where } k_j \text{ is a non negative integer } \forall j = 1, \ldots, k.$$

and $\mathcal{B}$ be the finite set of non negative integral cycles that read:

$$b = \sum_{j=1}^{j=k} b_j e_j \ \text{ with } \ 0 \leqslant b_j \leqslant n_j, \quad \forall j = 1, \ldots, k.$$

Any non negative integral cycle c in $S_2(AP_{\mathscr{P}}, \mathbb{R})$ reads:

$$c = l + b, \ \text{ where } \ l \in \mathscr{L}^+ \ \text{ and } \ b \in \mathcal{B}.$$

We have

$$\|c\| \leqslant \|l\| + \|b\| \leqslant \sum_{j=1}^{j=k} k_j \|\bar{e}_j\| + K_1$$

where

$$K_1 = \max_{b \in \mathcal{B}} \|b\|.$$

It follows that:

$$|\!|\!| c |\!|\!| \leqslant \sum_{j=1}^{j=k} k_j n_j |\!|\!| e_j |\!|\!| + K_1 \leqslant K_2 |l| + K_1,$$

where

$$K_2 = \max_{j=1,\ldots,k} |\!|\!| e_j |\!|\!|.$$

Since $|l| \leqslant |c|$ we conclude that there exists a positive constant $C_{\mathscr{P}}$ such that

$$|\!|\!| c |\!|\!| \leqslant C_{\mathscr{P}} \ |c|,$$

for any non negative integral cycle c in $H_2^+(AP_{\mathscr{P}}, \mathbb{R})$ and thus for any rational cycle in $H_2^+(AP_{\mathscr{P}}, \mathbb{R})$. □

Lemma 2.4. *The asymptotic Thurston semi-norm $|\!|\!| \cdot |\!|\!|$ is uniformly continuous over the rational cycles in $S_2(AP_{\mathscr{P}}; \mathbb{R})$.*

Proof. Let c' and c'' be two rational cycles in $S_2(AP_{\mathscr{P}}; \mathbb{R})$. For any integer $m > 0$ such that mc'' is an integral cycle, let $n(m)$ be the unique integer such that $mc'' - n(m)c'$ is a non negative cycle and $mc'' - (n(m) + 1)c'$ is not. This implies that there exists a vector f_i in the canonical basis of $\mathbb{R}^n$ such that:

$$\langle n(m)c', f_i \rangle \leqslant \langle mc'', f_i \rangle \leqslant \langle (n(m) + 1)c', f_i \rangle,$$

where $\langle \cdot, \cdot \rangle$ stands for the usual scalar product in $\mathbb{R}^n$. It follows that:

$$(\star) \qquad \lim_{m \to +\infty} \frac{n(m)}{m} = \frac{\langle c'', f_i \rangle}{\langle c', f_i \rangle}.$$

On the one hand, we have:

$$|||mc''||| \leqslant |||n(m)c'||| + |||mc'' - n(m)c'|||,$$

and thus

$$m|||c''||| - n(m)|||c'||| \leqslant |||mc'' - n(m)c'|||.$$

Since $n(m) \leqslant m$, we get

$$m|||c''||| - m|||c'||| \leqslant |||mc'' - n(m)c'||| \leqslant C_{\mathscr{P}}|mc'' - n(m)c'|.$$

Dividing by m and letting m go to $+\infty$ we get (using $(\star)$):

$$|||c''||| - |||c'||| \leqslant C_{\mathscr{P}} \frac{|\langle c', f_i \rangle c'' - \langle c'', f_i \rangle c'|}{\langle c', f_i \rangle} \leqslant 2C_{\mathscr{P}} \frac{|c' - c''|}{\langle c', f_i \rangle}.$$

Since there exists a positive constant ρ such that for every vector in the canonical basis of $\mathbb{R}^n$ and every cycle in $S_2(AP_{\mathscr{P}}, \mathbb{R})$, we have

$$0 < \rho \leqslant \langle c', f_i \rangle,$$

we finally get

$$|||c''||| - |||c'||| \leqslant \frac{2C_{\mathscr{P}}}{\rho} \cdot |c' - c''|.$$

On the other hand, by reversing the role of c' and c'' we also get

$$|||c'||| - |||c''||| \leqslant \frac{2C_{\mathscr{P}}}{\rho} \cdot |c' - c''|,$$

and thus

$$|\, |||c''||| - |||c'||| \,| \leqslant \frac{2C_{\mathscr{P}}}{\rho} \cdot |c' - c''|,$$

which proves the uniform continuity of the Thurston semi-norm. □

It follows from the above lemma that the function $c \mapsto |||c|||$ can be extended to a non negative continuous function defined on the whole simplex $S_2(AP_{\mathscr{P}}, \mathbb{R})$ we call it the *asymptotic Thurston map*. From the subadditivity of the map $c \mapsto |||c|||$ we get easily:

Lemma 2.5. *The asymptotic Thurston map is a bounded convex continuous non negative map.*

and

Corollary 2.1. *The set of zeros of the asymptotic Thurston map is a (possibly empty) convex subset of the simplex $S_2(AP_{\mathscr{P}}, \mathbb{R})$.*

3. Wang tilings

Let us first recall some basic definitions. A finite collection $\mathcal{W} = \{w_1, \dots, w_n\}$, where for $j = 1, \dots, n$, w_j is a unit square with sides parallel to the axes of $\mathbb{R}^2$ in the 2 dimensional Euclidean space $\mathbb{R}^2$, indexed by j and with colored edges, is called a collection of *Wang prototiles*. A Wang tiling is a tiling made with a finite collection $\mathcal{W}$ of Wang prototiles such that two adjacent squares share a same color on their adjacent edges. In 1966, R. Berger [4] gave a first example of a set of Wang prototiles that can tile the plane but cannot tile it periodically. This example was made with a collection of 20426 Wang prototiles. Since then, similar examples with a smaller set of Wang prototiles have been found. The state of the art is the example found by K. Culik [5] (see also [8]) made with 13 Wang prototiles and shown on Figure 2. Let us introduce two notions concerning Wang tilings that will

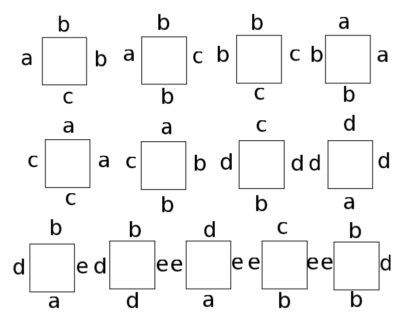

Figure 2. A collection of Wang prototiles.

be useful for the proof of our main theorem.

Forgetting colors:
For a given collection of Wang prototiles $\mathcal{W} = \{w_1, \dots, w_n\}$, we consider the set $\hat{\mathcal{W}} = \{\hat{w}_1, \dots, \hat{w}_n\}$, where, for $j = 1, \dots, n$, $\hat{w}_j$ is deduced from w_j by forgetting the colors on its sides and keeping its index j. It follows that $AP(\hat{\mathcal{W}})$ is a collection of n unit squares (indexed by j in $\{1, \dots, n\}$) glued respectively along their horizontal edges and their vertical edges.

Remark 3.1. $\Omega(\hat{\mathcal{W}})$ *is not empty and periodic orbits are dense in* $\hat{\Omega}(\mathcal{W})$*. Whenever* $\Omega(\mathcal{W}) \neq \emptyset$, $\Omega(\mathcal{W}) \subset \Omega(\hat{\mathcal{W}})$.

Enforcing colors:
Consider a collection of Wang prototiles $\mathcal{W} = \{w_1, \dots, w_n\}$ and fix $p > 0$. For each j in $\{1, \dots, n\}$ we consider the collection of tilings of the square $[-p-1/2, p+1/2]^2$ made with translated copies of prototiles in $\mathcal{W}$ so that colors of common edges of adjacent tiles coincide and the central tile that covers $[-1/2, +1/2]^2$ is a copy of w_j. We denote by $\{T^1_{w_j,p}, \dots, T^{l(w_i,\mathcal{W},p)}_{w_j,p}\}$ this collection of tilings.

For each tiling $T^l_{w_j,p}$ of $[-p-1/2, p+1/2]^2$ we consider the 4 colors:

- $\mathrm{Up}(T^l_{w_j,p})$ which is the restriction of $T^l_{w_j,p}$ to $[-p-1/2, p+1/2] \times [1/2, p+1/2]$;
- $\mathrm{Down}(T^l_{w_j,p})$ which is the restriction of $T^l_{w_j,p}$ to $[-p-1/2, p+1/2] \times [-p-1/2, -1/2]$;

- Left$(T^l_{w_j,p})$ which is the restriction of $T^l_{w_j,p}$ to $[-p-1/2,-1/2]\times[-p-1/2,p+1/2]$;
- Right$(T^l_{w_j,p})$ which is the restriction of $T^l_{w_j,p}$ to $[1/2,p+1/2]\times[-p-1/2,p+1/2]$.

and associate the Wang prototile $w_{i,l}$ whose index is the pair (j,l) and whose edges inherit the color :

- Up$(T^l_{w_j,p})$ for the top edge;
- Down$(T^l_{w_j,p})$ for the bottom edge;
- Left$(T^l_{w_j,p})$ for the left edge;
- and Right$(T^l_{w_j,p})$ for the right edge.

We denote by $\mathcal{W}^p$ the collection of Wang prototiles $w_{j,l}$ when j runs from 1 to n and l from 1 to $l(w_i,\mathcal{W},p)$.

Remark 3.2. *$\mathcal{W}$ tiles the plane if and only if , for each $p>1$, $\mathcal{W}^p$ (and thus $AP(\mathcal{W}^p)$) is well defined.*

The importance of Wang tilings stems from the following result proved by L. Sadun and R. Williams (which is in fact valid in any dimension).

Theorem 3.3. [11] *For any finite collection of prototiles $\mathscr{P}$ that tiles the plane, there exists a finite collection of Wang prototiles $\mathcal{W}$ such that the dynamical systems $(\Omega_{\mathscr{P}},\mathbb{R}^2)$ and $(\Omega_{\mathcal{W}},\mathbb{R}^2)$ are orbit equivalent.*

Remark 3.4. *Actually the homeomorphism that realizes the orbit equivalence of Theorem 3.3 possesses some important rigidity properties that will be detailed in the next section.*

4. Reduction to Wang tilings

Lemma 4.1. *Theorem 1.1 is true if it is true for any finite collection of Wang prototiles.*

Proof. The proof splits in the proof of 3 claims.
Claim 1:
Theorem 1.1 is true if it is true for any finite collection of rational prototiles.
Proof of Claim 1: In order to prove this claim we need, as announced in Remark 3.4, to go deeper in the proof of Sadun and Williams of Theorem 3.3 in [11] and, for the sake of convenience, we sketch the construction given therein in full details. Consider a finite collection of prototiles $\mathscr{P}=\{p_1,\ldots,p_n\}$ that tiles the plane. For any $\epsilon>0$, one can construct a finite collection of rational prototiles $\mathscr{P}'=\{p'_1,\ldots,p'_n\}$ such that:

- Each p'_i has the same number of edges as p_i, and the p_i's are ϵ-close to the p_i's for the Haussdorff distance and the corresponding edges have the same colors;
- The Anderson-Putnam complex $AP_{\mathscr{P}'}$ is homeomorphic to $AP_{\mathscr{P}}$.

This amounts to solve finitely many equations with integral coefficients and to use the fact that for such systems of equations, rational solutions are dense in the set of solutions. The one-to-one correspondence of the 2-cells of $AP_{\mathscr{P}'}$ and $AP_{\mathscr{P}}$ yields the natural identifications:

$$H_2^+(AP_{\mathscr{P}'},\mathbb{R}^2)=H_2^+(AP_{\mathscr{P}},\mathbb{R}^2)\quad\text{and}\quad S_2(AP_{\mathscr{P}'},\mathbb{R}^2)=S_2(AP_{\mathscr{P}},\mathbb{R}^2).$$

On the one hand, the construction of the asymptotic Thurston norm on $S_2(AP_{\mathscr{P}'}, \mathbb{R}^2)$ coincides with the similar construction on $S_2(AP_{\mathscr{P}}, \mathbb{R}^2)$. On the other hand, there is a natural cone isomorphism $\mathcal{I}$ between $\Theta(\Omega_{\mathscr{P}'})$ and $\Theta(\Omega_{\mathscr{P}})$ which is defined by:

$$\mathcal{I}(\mu)(A) = \sum_{i=1}^{i=n} \mu(\pi^{-1}(p_i) \cap A)) \cdot \frac{\lambda(p_i)}{\lambda(p_i')},$$

for any mesurable set A in $\Omega_{\mathscr{P}}$ and any measure μ in $\Theta(\Omega_{\mathscr{P}})$. It follows easily that

$$\mathrm{Ev}(\mathcal{I}(\mu)) = \mathrm{Ev}(\mu), \quad \forall \mu \in \Theta(\Omega_{\mathscr{P}}).$$

This proves Claim 1.
Let us illustrate the above construction on the classical example of Penrose tilings. Consider the 'thin' and 'fat' triangles displayed in Figure 3 [4]. Together with their rotation by multiples of $2\pi/10$, they generate a set of prototiles $\mathscr{P}$ with 40 elements which in turn, generates the Penrose dynamical system $(\Omega_{\mathscr{P}}, \mathbb{R}^2)$.

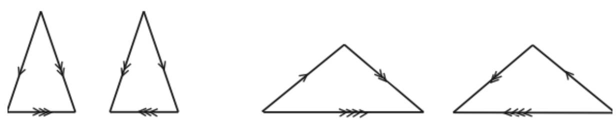

FIGURE 3. The tiles of the Penrose tiling.

Figure 4 shows a patch in $\mathbb{R}^2$ tiled by Penrose prototiles.

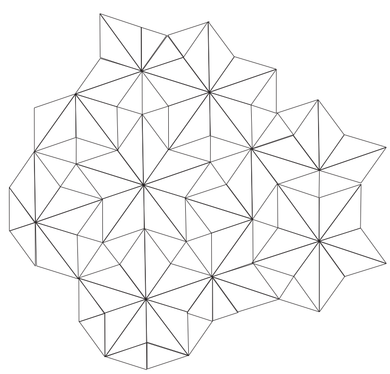

FIGURE 4. A patch of a Penrose tiling.

Figure 5 shows now the same patch (up to homeomorphism) tiled with rational prototiles.

Claim 2:
Theorem 1.1 is true for any finite collection of rational prototiles if it is true for any finite collection of integral prototiles.
Proof of Claim 2: The next step in [11] is to transform a finite collection of rational prototiles $\mathscr{P}'$ into a finite collection of integral prototiles. Using an homothety with

[4] It is customary to use arrowheads to indicate adjacency rules. Each triangle can be represented as a polyhedron by replacing the arrowheads by appropriate dents and bumps to fit the general definition of tilings given above.

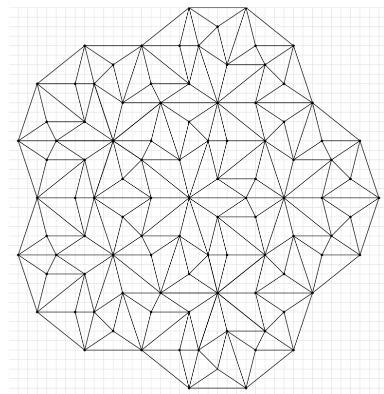

Figure 5. A patch of the rational Penrose tiling.

an integral dilatation factor, one can transform the collection $\mathscr{P}' = \{p'_1, \dots, p'_n\}$ in a family of integer prototiles $\mathscr{P}'' = \{p''_1, \dots, p''_n\}$. Clearly both dynamical systems $(\Omega_{\mathscr{P}'}, \mathbb{R}^2)$ and $\Omega_{\mathscr{P}''}, \mathbb{R}^2)$ are orbit equivalent, the homeomorphism that realizes this equivalence maps translation invariant measures onto translation invariant measures and the two Anderson-Putnam complex $AP_{\mathscr{P}'}$ and $AP_{\mathscr{P}''}$ are homothetic. This proves Claim 2.

Claim 3:
Theorem 1.1 is true for any finite collection of integral prototiles if it is true for any finite collection of Wang prototiles.
Proof of Claim 3: One proceeds in 2 steps:

1 One replaces the straight edges of the prototiles in $\mathscr{P}''$ with zig-zags, that is with sequences of unit displacements in the coordinates directions. We denote by $\widehat{\mathscr{P}} = \{\hat{p}_1, \dots, \hat{p}_n\}$ the new collection of prototiles obtained this way. Figure 6 shows how, in the particular case of the Penrose collection of rational prototiles, the patch described in Figure 5 is transformed.

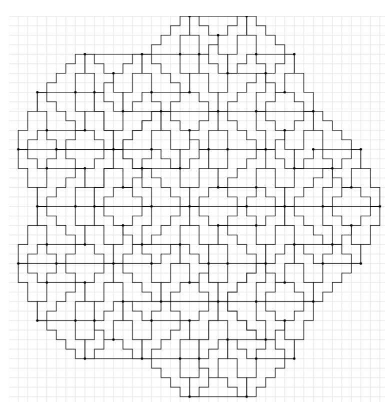

Figure 6. A patch of a square Penrose tiling.

2 It remains to put a label and appropriate colors on the edges of each square in each prototile in $\widehat{\mathscr{P}}$ to obtain a Wang tiling. The encoding is made in such a way that each edge of a square which is in the interior of a prototile of $\widehat{\mathscr{P}}$ forces as neighbors only the square which is its neighbor in the prototile and that any edge which meets the boundary of a prototile in $\widehat{\mathscr{P}}$ has its color given by the one of the prototile it belongs to. We denote by $\mathcal{W}$ the finite collection of Wang prototiles obtained with this construction.

It follows that the dynamical systems $(\Omega_{\mathcal{W}}, \mathbb{R}^2)$ and $(\Omega_{\widehat{\mathscr{P}}}, \mathbb{R}^2)$ are conjugate and that both Anderson-Putnam complexes $AP_{\mathcal{W}}$ and $AP_{\widehat{\mathscr{P}}}$ are homeomorphic. This allows us to identify:

$$H_2^+(AP_{\widehat{\mathscr{P}}}, \mathbb{R}^2) = H_2^+(AP_{\mathcal{W}}, \mathbb{R}^2) \quad \text{and} \quad S_2(AP_{\widehat{\mathscr{P}}}, \mathbb{R}^2) = S_2(AP_{\mathcal{W}}, \mathbb{R}^2).$$

and proves Claim 3. □

5. Proof of Theorem 1.1

Thanks to Lemma 4.1 we are reduced to prove Theorem 1.1 in the particular case of a finite collection of Wang prototiles $\mathcal{W} = \{w_1, \ldots, w_n\}$.

• **Assume first that $\Omega_{\mathcal{W}} \neq \varnothing$.**

This implies that the set of translation-invariant probability measures $\Theta(\Omega_{\mathcal{W}}) \neq \varnothing$. Consider an ergodic measure $\mu \in \Theta(\Omega_{\mathcal{W}})$. From the Birkhoff Ergodic Theorem, we know that for μ-almost every tiling T in $\Omega_{\mathcal{W}}$ and for every prototile w_i in $\mathcal{W}$:

$$\lim_{p\to+\infty} \frac{1}{(2p+1)^2}\, \mathcal{N}(w_i, p) = \mu(\pi^{-1}(w_i)),$$

where $\mathcal{N}(w_i, p)$ stands for the number of copies of w_i that appear in T in the square $[-1/2 - p, p + 1/2]^2$. Fix $p > 0$ and consider the periodic tiling $\widehat{T}_p$ in $\Omega_{\widehat{\mathcal{W}}}$ obtained from T by repeating the pattern of T in $[-1/2 - p, p + 1/2]^2$. More precisely for any (q, r) in $\mathbb{Z}^2$, the tile of $\widehat{T}_p$ centered at (q, r) corresponds to the same prototile as the tile centered at $(q - m(2p + 1), r - n(2p + 1))$ where (m, n) are chosen so that $(q - m(2p+1), r - n(2p+1)) \in [-1/2 - p, p + 1/2]^2$. Consider the probability measure $\hat{\mu}_p$ which is equidistributed along the $\mathbb{R}^2$-orbit of the tiling $\widehat{T}_p$.

On the one hand, notice that $\mathrm{Ev}(\hat{\mu}_p)$ is a cycle in $S_2(AP_{\widehat{\mathcal{W}}}, \mathbb{R})$ which is given by:

$$\mathrm{Ev}(\hat{\mu}_p) = \sum_{i=1}^{i=n} \hat{\mu}_p(\pi^{-1}(\hat{w}_i))\widehat{w}_i = \frac{1}{(2p+1)^2} \sum_{i=1}^{i=n} \mathcal{N}(w_i, p)\widehat{w}_i.$$

It follows that:

$$\lim_{p\to+\infty} \mathrm{Ev}(\hat{\mu}_p) = \sum_{i=1}^{i=n} \mu(\pi^{-1}(w_i))\widehat{w}_i.$$

On the other hand, the natural inclusion $\Omega_{\mathcal{W}} \subset \Omega_{\widehat{\mathcal{W}}}$ allows us to see the measure μ as a measure $\hat{\mu}$ in $\Theta(\Omega_{\widehat{\mathcal{W}}})$ and the cycle $\mathrm{Ev}(\mu) = \sum_{i=1}^{i=n} \mu(\pi^{-1}(w_i))w_i$ in $S_2(AP_{\mathcal{W}}, \mathbb{R})$ can be identified (through the above inclusion) with the cycle $\mathrm{Ev}(\hat{\mu}) = \sum_{i=1}^{i=n} \mu(\pi^{-1}(w_i))\widehat{w}_i$ in $S_2(AP_{\widehat{\mathcal{W}}}, \mathbb{R})$. Thus:

$$\lim_{p\to+\infty} \mathrm{Ev}(\hat{\mu}_p) = \mathrm{Ev}(\hat{\mu}).$$

Since the $\mathbb{R}^2$-orbit of the tiling $\widehat{\mathcal{W}}_p$ is a 2-torus embedded in $\Omega_{\widehat{\mathcal{W}}}$, it follows directly that $\| \mathrm{Ev}(\hat{\mu}_p)\| = 0$. The continuity of the Thurston semi-norm (Lemma 2.4) implies that $\| \mathrm{Ev}(\hat{\mu})\| = 0$ and thus $\| \mathrm{Ev}(\mu)\| = 0$. We conclude that $H_2^+(AP_{\mathcal{W}}, \mathbb{R}) \neq \varnothing$ and the set of zeros of the asymptotic Thurston semi-norm on $S_2(AP_{\mathcal{W}}, \mathbb{R})$ is not empty and contains $\mathrm{Ev}(\Theta(\Omega_{\mathcal{W}}))$.

• **Assume now $H_2^+(AP_{\mathcal{W}}, \mathbb{R}) \neq 0$ and that the Thurston semi-norm has a zero in $S_2(AP_{\mathcal{W}}, \mathbb{R})$.**

Let $c \in S_2(AP_{\mathcal{W}}, \mathbb{R})$ be such that $\|c\| = 0$. The continuity of the Thurston semi-norm and the density of rational cycles in $S_2(AP_{\mathcal{W}}, \mathbb{R})$, implies that for each sequence of rational cycles $(c_l)_{l\geqslant 0}$ in $S_2(AP_{\mathcal{W}}, \mathbb{R})$ such that $\lim_{l\to+\infty} c_l = c$, there exists a sequence of integers $(n_l)_{l\geqslant 0}$ such that $n_l c_l$ is an integral cycle and a sequence of surfaces $(\mathcal{F}_l)_{l\geqslant 0}$ such that for each $l \geqslant 0$:

$$\mathcal{F}_l \in [c_l] \quad \text{and} \quad \lim_{l\to+\infty} \frac{|\chi(\mathcal{F}_l)|}{n_l} = 0.$$

Fix now $p > 0$ and, for each l big enough, consider the surface (with boundary) $\mathcal{F}_{l,p}$ which is made of all the Wang tiles of $\mathcal{F}_l$ that are at the center of the square $[-1/2 - p, p + 1/2]^2$ embedded in $\mathcal{F}_l$ (that is to say, these tiles which are at a distance larger than p from a singular point). Let $c_{l,p}$ be the chain associated to $\mathcal{F}_{l,p}$ in $C_2(AP_{\mathcal{W}}, \mathbb{R})$. There exists a constant $K > 0$ such that, for l big enough,

$$|c_{l,p} - n_l c_l| \leqslant K\ |\chi(\mathcal{F}_l)|\ p^2$$

and thus

$$\lim_{l\to+\infty} \frac{c_{l,p}}{|c_{l,p}|} = c.$$

Clearly W^p is not empty since when l is big enough, $\mathcal{F}_{l,p}$ is not empty. From the very construction of $\mathcal{F}_{l,p}$ we get that the chain $c_{l,p}$ is the image of a 2-chain ${c_{l,p}}^{(p)}$ in $C_2(AP(\mathcal{W}^p), \mathbb{R})$ through the canonical projection $\pi^{(p)} : C_2(AP(\mathcal{W}^p), \mathbb{R}) \to C_2(AP(\mathcal{W}), \mathbb{R})$. Let $c^{(p)}$ be an accumulation point $C_2(AP(\mathcal{W}^p), \mathbb{R})$ of the sequence of normalized chains $\left(\frac{c_{l,p}^{(p)}}{|c_{l,p}^{(p)}|}\right)_{l>0}$. We easily check that

$$|\partial {c_{n,l}}^{(p)}| \leqslant K|\chi(\mathcal{F}_l)|p^2,$$

which implies that $c^{(p)}$ is a non negative 2-cycle in $H_2^+(CW(\mathcal{W}^p), \mathbb{R})$ and that $\pi^p(c^{(p)}) = c$. It follows that

$$c \in \pi^{(p)}(H_2^+(AP(\mathcal{W}^p, \mathbb{R}))),\ \forall p > 0.$$

Since it is well known (see [3] or [2]) that

$$\Theta(\Omega_{\mathcal{W}}) = \bigcap_{p>0} H_2^+(AP(\mathcal{W}^p, \mathbb{R}))$$

we deduce that $\mathcal{W}$ tiles the plane and that that zeros of the Thurston semi-norm on $S_2(AP_{\mathcal{W}}, \mathbb{R})$ are contained in $\Theta(\Omega_{\mathcal{W}})$.

This ends the proof of our main Theorem.

6. Discussion and examples

Let us examine the different situations that may occur according to the family $\mathcal{W}$ of Wang prototiles we consider. One extreme situation is when $\mathcal{W}$ does not tile the plane, in this case either $H_2^+(AP_\omega, \mathbb{R}) = \emptyset$ or $H_2^+(AP_\omega, \mathbb{R}) \neq \emptyset$ and the Thurston semi-norm remains strictly positive in this cone. The other extreme situation is when the colors of the edges are forgotten. In this situation, the periodic orbits are dense in $\Omega_{\mathcal{W}}$, $\mathrm{Ev}(\Theta(\Omega_{\mathcal{W}}))$ is the whole simplex $S_2(AP_{\mathcal{W}}, \mathbb{R})$ and the Thurston semi-norm vanishes on the whole simplex.
Let us concentrate now on the case when $H_2^+(AP_{\mathcal{W}}, \mathbb{R}) \neq \emptyset$ and the (convex) set of zeros of the asymptotic Thurston semi-norm (which coincides with $\mathrm{Ev}(\Theta(\Omega_{\mathcal{W}}))$) is not empty. Different cases may occur:

- $\mathrm{Ev}(\Theta(\Omega_{\mathcal{W}}))$ is reduced to a single cycle c.
  - If c is not rational, then $\mathcal{W}$ cannot tile the plane periodically. This is exactly the situation we studied earlier for the Penrose tiling.
  - If c is rational,
    * either $\|c\| = 0$ which means that $\mathcal{W}$ can tile the plane periodically;
    * or $\|c\| \neq 0$ and $\mathcal{W}$ cannot tile the plane periodically: this is exactly what happens for the Robinson set of Wang prototiles [9].
- $\mathrm{Ev}(\Theta(\Omega_{\mathcal{W}}))$ is not reduced to a single cycle. In this case we are left with a series of questions, for instance:
  - **Question 1**: Can we find $\mathcal{W}$ such that rational cycles are not dense in $\mathrm{Ev}(\Theta(\Omega_{\mathcal{W}}))$?
  - **Question 2**: When does $\mathrm{Ev}(\Theta(\Omega_{\mathcal{W}}))$ contain a ball in $S_2(AP_{\mathcal{W}}, \mathbb{R})$?

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